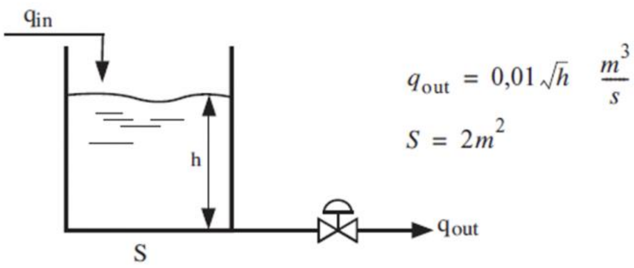
Series 3 solutions

Exercise 1



At $t = 0$, while the system is in a stationary state $q_{in} = 0{,}015 m^2/s$, the feed flow is cut..

- At a stationary state: $\bar{q}_{in} = \bar{q}_{out}$

Therefore: $0{,}015\left[\frac{m^3}{s}\right] = 0{,}01\sqrt{\bar{h}}\left[\frac{m^3}{s}\right] \Rightarrow \bar{h} = 2{,}25[m]$

- Mass balance:

$$\frac{dm}{dt} = \frac{d(\rho A)}{dt} = m_{in} - m_{out}$$

$$\rho S\frac{dh}{dt} = \rho q_{in} - \rho q_{out} = \rho q_{in} - \rho(0{,}01\sqrt{h})$$

$$S\frac{dh}{dt} = q_{in} - 0{,}01\sqrt{h}$$

a) Empty half of the reservoir

For $t \geq 0$, $q_{in} = 0$. Therefore,

$$\frac{dh}{dt} = -0{,}01\sqrt{h}$$

By integrating the equation by separation of variables:

$$\int_{\bar{h}}^{\frac{\bar{h}}{2}} \frac{dh}{\sqrt{h}} = -\frac{0{,}01}{S}\int_0^T dt$$

With $S = 2[m^2]$ et $\bar{h} = 2{,}25[m]$, we get: $T = 176[s]$

b) Empty the entire reservoir

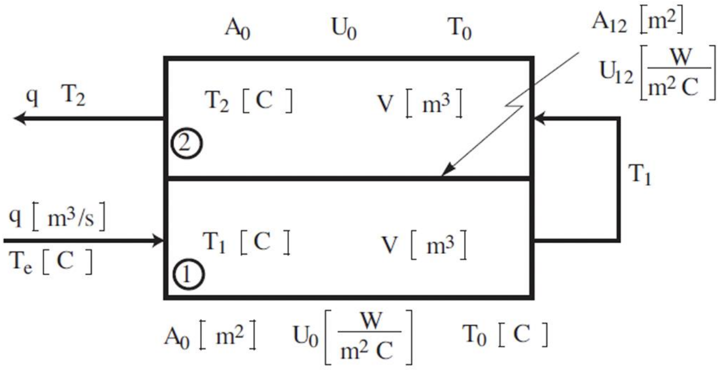
$$\int_{\bar{h}}^{0} \frac{dh}{\sqrt{h}} = -\frac{0{,}01}{S}\int_0^T dt \implies T = 600[s]$$

<u>*Note*</u>: if the flow rate q_{out} were proportional to h,i.e. $q_{out} = kh$, then:

$$S\frac{dh}{dt} = -kh \implies \int_{\bar{h}}^{h} \frac{dh'}{\sqrt{h'}} = -\frac{k}{S}\int_0^t dt' \implies h(t) = \bar{h}\cdot e^{-\frac{k}{S}t}$$

Therefore, for $h(t) \to 0$, $t \to \infty$.

Exercise 2



a) Dynamic model for the system

Heat balance for zone 1:

$$V\rho c_p \frac{dT_1}{dt} = q\rho c_p(T_e - T_1) - U_{12}A_{12}(T_1 - T_2) - U_0A_0(T_1 - T_0)$$

Heat balance for zone 2:

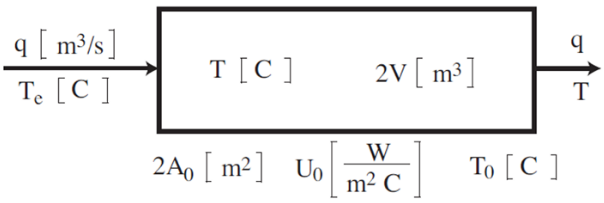
$$V\rho c_p \frac{dT_2}{dt} = q\rho c_p(T_1 - T_2) - U_{12}A_{12}(T_2 - T_1) - U_0A_0(T_2 - T_0)$$

b) Linearity

In each differential equation, the product qT is appearing:

- If q varies, representing for instance the entry variable of the system: nonlinear model
- If q is constant: linear model

c) Comparison of the two systems



Heat balance for case 2 (only one zone):

$$2V\rho c_p \frac{dT}{dt} = q\rho c_p(T_e - T) - 2U_0A_0(T - T_0)$$

Combined heat balance for case 1 (two zones):

$$V\rho c_p \left(\frac{dT_1}{dt} + \frac{dT_2}{dt}\right) = q\rho c_p(T_e - T_1) - 2U_0A_0\left(\frac{T_1 + T_2}{2} - T_0\right)$$

Thus,

- the two systems are not equivalent,
- case 1 corresponds to case 2 for $T_1 = T_2 = T$.

d) Nonhomogeneous sections

If the two zones are not homogeneous, the system will have spatially varying parameters (described by partial differential equations)

Exercise 3

$$\dot{x}_1 = x_1 + x_2 - (u_1 + u_2) \qquad x_1(0) = 1$$
$$\dot{x}_2 = x_1^2 - (x_2 - 1)^2 + x_1 x_2 - u_1^2 - u_2 \qquad x_2(0) = 1$$

$$y_1 = x_1(1 + x_2) + u_1$$
$$y_2 = x_1 + x_2 - u_2$$

Linearize this model for the equilibrium point corresponding to $\bar{u}_1 = \bar{u}_2 = 1$, for positive values of $\bar{x}_1$ and $\bar{x}_2$.

At the equilibrium point corresponding to $\bar{u}_1 = \bar{u}_2 = 1$, the variables $\bar{x}_1$ and $\bar{x}_2$ satisfy the relations:

$$0 = \bar{x}_1 + \bar{x}_2 - 2$$
$$0 = \bar{x}_1{}^2 - (\bar{x}_2 - 1)^2 + \bar{x}_1 \bar{x}_2 - 2$$

The solution to this system of nonlinear equations for positive values of $\bar{x}_1$ and $\bar{x}_2$ is:

$$\bar{x}_1 = \bar{x}_2 = 1$$

By linearizing a nonlinear dynamic system around this point, the following matrices A, B, C and D are obtained:

$$A = \begin{bmatrix} 1 & 1 \\ 2\bar{x}_1 + \bar{x}_2 & -2(\bar{x}_2 - 1) + \bar{x}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -1 \\ -2\bar{u}_1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 + \bar{x}_2 & \bar{x}_1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

With:

$\delta\dot{x} = A\delta x + B\delta u$ and $\delta y = C\delta x + D\delta u$ where $\delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$ etc.